THEOREM 4 An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

DEFINITION Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

EXAMPLE 3 Let A be an invertible $n \times n$ matrix—say, $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem. ■

EXAMPLE 4 Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix, I_n. That is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

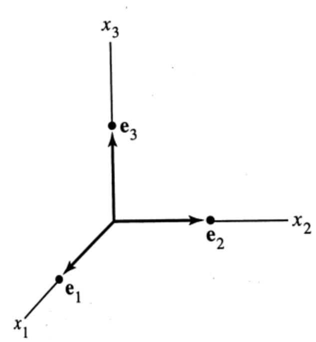


FIGURE 1
The standard basis for $\mathbb{R}^3$.

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

EXAMPLE 6 Let $S = \{1, t, t^2, \ldots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.

EXAMPLE 7 Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}, \quad \text{and} \quad H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $\text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$. Then find basis for the subspace H.

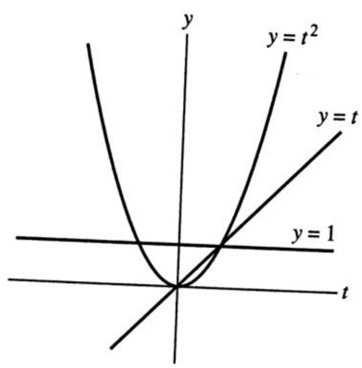
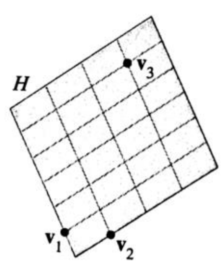


FIGURE 2
The standard basis for $\mathbb{P}_2$.



THEOREM 5 **The Spanning Set Theorem**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

EXAMPLE 8 Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE 9 It can be shown that the matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix B in Example 8. Find a basis for Col A.

THEOREM 6 The pivot columns of a matrix A form a basis for Col A.

PRACTICE PROBLEMS

1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

4. Let V and W be vector spaces, let $T : V \to W$ and $U : V \to W$ be linear transformations, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis for V. If $T(\mathbf{v}_j) = U(\mathbf{v}_j)$ for every value of j between 1 and p, show that $T(\mathbf{x}) = U(\mathbf{x})$ for every vector $\mathbf{x}$ in V.

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A single vector by itself is linearly dependent.

b. If $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

c. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

d. A basis is a spanning set that is as large as possible.

e. In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

22. a. A linearly independent set in a subspace H is a basis for H.

b. If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

c. A basis is a linearly independent set that is as large as possible.

d. The standard method for producing a spanning set for Nul A, described in Section 4.2, sometimes fails to produce a basis for Nul A.

e. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.